Halit Eren. "Optimal Control."

Optimal Control

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98.1 Introduction

Optimal control maximizes (or minimizes) the value of a function chosen as the **performance index** or *cost function* of an operational control system. Optimal control theory, on the other hand, is the mathematics of finding parameters that cause the performance index to take an extreme value subject to system constraints.

Optimal control is applied in many disciplines, such as satellites and aerospace, aircraft and spacecraft, chemical engineering, communications engineering, robots and robotics, power systems, electric drives, computers and computer systems, etc. In many applications simple interconnections of control devices and controllers do not provide the most economic operation for which optimal control offers the solution. Hence, **optimization** is useful in obtaining the best results from a known process. If the process is operating under a steady-state condition, optimization considers the process stationary, and it is concerned only with the operating points. When the process is stationary, the resulting optimum operating point can easily be maintained by setting the predetermined set points and precalculated control parameters. Nevertheless, if the process changes from time to time, new optimum set points for the system need to be determined for each change.

The performance of a system is optimized for many reasons, such as improving the quality, increasing production, decreasing waste, obtaining greater efficiency, maximizing the safety, saving time and energy, and so on. In many optimization problems, boundary conditions are imposed by the system for safety in operations, availability of minimum and maximum power, limitations in storage capacity, capability of the operating machinery, temperature, speed, force, acceleration, indeed, for any other hard physical reason.

For the solution of optimal control problems many different methods may be used depending on the nature of the problem. For example, if the performance index and constraints can be formulated as linear algebraic functions of the controlled variables, then selection of the linear programming may be the best way to go. Simplex methods may provide a good way of solving the linear programming problem. If the equations describing the system are nonlinear, then the solution may involve nonlinear techniques or linearization of the problem in some subregions. If the problem involves determination of the largest value of a function with two or more variables, then the steepest ascent (or descent) method, sometimes termed *hill climbing* or the *gradient method* may be used.

Hill-climbing methods are still popularly used and easy to understand. They are briefly discussed here, first, to lay a good background in the sound understanding of optimal control theory.
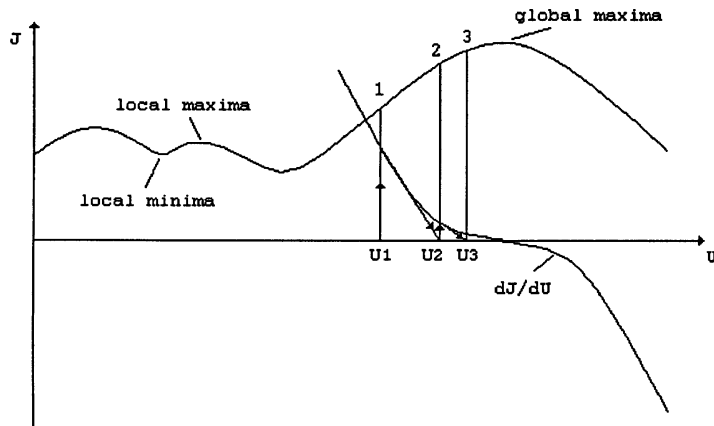
FIGURE 98.1   Hill-climbing technique illustrated in the region of absolute maxima. The sign of the first derivative of the function changes as the derivative passes through the maximum point. This figure also demonstrates the local maxima and minima that should be taken care optimal control design.

(a)   (b)
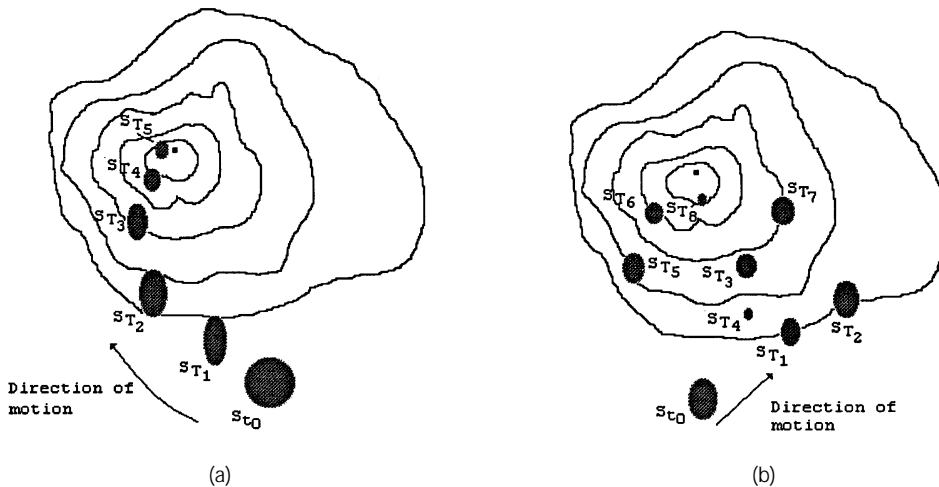
FIGURE 98.2   Examples of hill climbing method, (a) gradient, (b) random walk. All these methods use an initial estimated point where the first derivative is guessed to be zero. This point is then used, in the direction of steepest ascent or descent as the case may be, for the next point until the absolute maximum or minimum is found.

An illustration of the hill-climbing technique in the region of absolute minima is given in Fig. 98.1. This figure also illustrates a common pitfall in control theory, the local maxima and minima. As can be seen in this figure, the sign of the first derivative of the function changes as the derivative passes through the maximum point. This method commonly uses an initial estimated point where the first derivative is considered (guessed) to be zero. This point is then used, in the direction of steepest ascent or descent as the case may be, for the next point until the absolute maxima or minima is found without being trapped in local extreme points. There are many different methods to find maxima or minima by hill-climbing methods and Fig. 98.2 illustrates some examples of implementation.

Apart from hill-climbing techniques, there are many new methods available for the solution of modern optimal control problems involving hyperspaces (many variables) complex in nature and having stringent constraints. If a good mathematical model of the system is available, optimization methods may be used for finding the optimum conditions on the model, instead of seeking optimization on the actual process. This indicates that availability of the model determines the type of optimization method to be employed.

98.2 Cost Function

In designing optimal control systems, rules may be set for determining the control decisions, subject to certain constraints, such that some measure of deviations from an ideal case is minimized. That measure is usually provided by a selected performance index or cost function. Performance index is a function whose value indicates how well the actual performance of the system matches the desired performance. Appropriate selection of the performance index is important since it determines the nature and complexity of the problem. That is, if the optimal control is linear or nonlinear, stationary or time-varying will be dependent on the selected performance index. The choice of the performance index depends on system specifications, physical realizability, and the restrictions on the controls to be used. In general, the choice of an appropriate performance index involves a compromise between a meaningful evaluation of system performance and availability of feasible mathematical descriptions.

The performance index is selected by the engineer to make the system behave in a desired fashion. By definition, a system whose design minimizes the selected performance index is optimal. It is important to point out that a system optimal for one performance index may not be optimal under another performance index. In practical systems, due to possible complexities and cost of implementation, it may be better to employ approximate optimal control laws which are not rigidly tied to a single performance index.

Before starting optimization of any system, it is necessary to formulate the system by having information on system parameters and describing equations, constraints, class of allowable control vectors, and the selected performance indexes. Then the solution can proceed by determining the optimal control vector $\mathbf{u}(k)$ within the class of allowable vectors. The control vector $\mathbf{u}(k)$ will be dependent on such factors as the nature of the performance index, constraints, initial values of state, initial outputs, and the desired state as well as the desired outputs. If analytical solutions are impossible or too complicated, then alternative computational solutions may be employed.

In simple cases, errors between the desired and actual responses can be chosen as the performance index to be minimized. As an example, different descriptions of error between actual and desired responses are depicted in Fig. 98.3. The aim is to keep the errors as small as possible. The time integral of error gives the severity of the error. However, since the positive and negative errors mathematically cancel each other, absolute values must be used:

$$\text{Integral Absolute Error} = \text{IAE} = \int_0^T \left| e(\tau) \right| d\tau \tag{98.1}$$
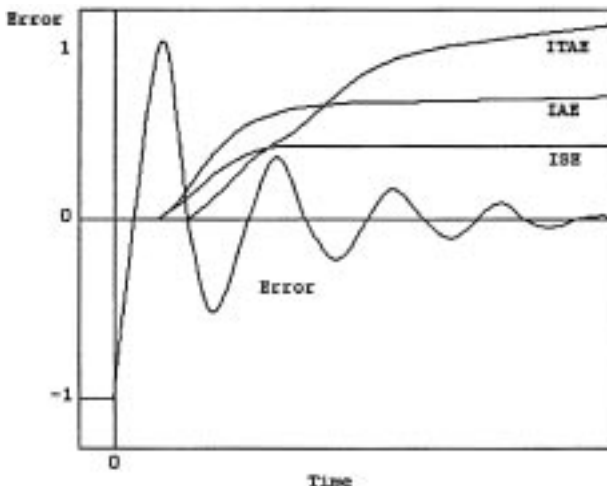
FIGURE 98.3 Description of errors between actual and desired responses. In simple optimal control applications, errors between the desired and actual responses can be chosen as the performance indexes. The aim is to keep the errors as small as possible.

In many cases it is better to use integral squared errors which take into account large errors rather than regular ones:

$$\text{Integral Squared Error} = \text{ISE} = \int_0^T e^2(\tau)d\tau \qquad (98.2)$$

Time-dependent functions can be implemented to capture errors occurring late in time rather than normal transients:

$$\text{Integral Time Absolute Error} = \text{ITAE} = \int_0^T \left| e(\tau) \right| \tau d\tau \qquad (98.3)$$

and

$$\text{Integral Time Squared Error} = \text{ITSE} = \int_0^T e^2(\tau)\tau d\tau \qquad (98.4)$$

Mean square error, MSE, is also commonly used since slight modification leads to inclusion of statistical techniques and analysis of random noise:

$$\text{Mean Square Error} = \text{MSE} = \lim_{T \to \infty} \int_0^T 1/Te^2 \tau d\tau \qquad (98.5)$$

Modern optimal control theory is developed within **state-space** framework and performance indexes are more complex and comprehensive as explained below.

Suppose that the control command of a system is expressed in vectorial form as $\mathbf{u}$ and the state of the system is described by $\mathbf{x}$ (*Oxford English Dictionary* defines *state* as condition with respect to circumstances, attributes, structure, form phase or the like). Further, suppose that the rate of chance of state $\dot{\mathbf{X}}$ is a function of state $\mathbf{x}$, control command $\mathbf{u}$, and time t

$$\dot{\mathbf{X}} = f(\mathbf{x}, \mathbf{u}, t) \quad \mathbf{x}(0) = \mathbf{x}_0 \text{ known} \qquad (98.6)$$

Then a control law $\mathbf{u}(\mathbf{x}, t)$ or a control history $\mathbf{u}(t)$ is determined such that a performance index or a scalar functional

$$J(u) = \int_0^T g\big(\mathbf{x}(\tau), \mathbf{u}, \tau\big) d\tau \qquad (98.7)$$

takes a minimum value out of all other possibilities and still holds Eq. 98.6. A boundary relationship $\mathbf{x}(T) = \mathbf{x}_f$ must also be met as a constraint.

A most common form of $J(u)$ is the minimum time control in which

$$J(u) = \int_0^T d\tau = T \qquad (98.8)$$

Many different criteria are also used, such as minimum fuel, minimum energy, and other quadratic forms

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$$J(u) = \int_0^T \left| u(\tau) \right| d\tau \tag{98.9}$$

$$J(u) = \int_0^T u^2(\tau) d\tau \tag{98.10}$$

$$J(u) = \int_0^T \left(qx^2(\tau) + ru^2(\tau) \right) d\tau \tag{98.11}$$

A general term for continuous time performance index leading to optimal control is expressed as:

$$J\left(u(t) \right) = \int_0^T g\left(x(\tau), u(\tau) \right) d\tau \tag{98.12}$$

This performance index is minimized for the constraints

$$\dot{\mathbf{X}}(t) = f\left(\mathbf{x}(t), \mathbf{u}(t), t \right) \quad \text{for } t \in \left(t_0, t_f \right) \tag{}$$

and $\mathbf{x}(t)$ is an admissible state, $\mathbf{x}(t) \in \mathbf{X}(t), \ \forall \ t \in \ (t_0, t_f)$ are satisfied.

Slight variations of Eqs. 98.9 to 98.12 lead to mathematics of the discrete time or digital versions of optimal control.

98.3   Calculus of Variations

Calculus variations are suitable for solving linear or nonlinear optimization problems with linear or nonlinear boundary conditions. Basically, it is a collection of many different analytical methods and they are discussed differently from book to book. Here, a typical approach which leads to more general and widely used modern theories is introduced.

Consider a dynamic system operating in a time interval $t_0 < t < t_f$

$$\dot{\mathbf{X}} = f\left[\mathbf{x}, \mathbf{u}, t \right] \tag{98.13}$$

where the initial state $\mathbf{x}_0$ is given.

The system has n state and m control variables. The scalar function to be optimized is

$$J = k\left[\mathbf{x}_f, t_f \right] + \int \mathbf{L}\left[\mathbf{x}, \mathbf{u}, t \right] dt \tag{98.14}$$

Define a scalar, such as Hamiltonian

$$H = H\left[\mathbf{x}, \mathbf{u}, \lambda t \right] = L\left[\mathbf{x}, \mathbf{u}, t \right] + \sum \lambda_i f_i \tag{98.15}$$

λ is known as the **Lagrange multiplier**. Also, define a modified objective function:

$$J = k\left[\mathbf{x}_1, t_1 \right] + \int H dt \tag{98.16}$$

the resulting solution is optimal when

$$\lambda = -\delta H \backslash \delta x = -\delta L \backslash \delta x - \sum \lambda_i \delta f_i \backslash \delta x \qquad (98.17)$$

$$\lambda_f = \delta k \backslash \delta x \qquad (98.18)$$

$$\delta H \backslash \delta u = \delta L \backslash \delta u + \sum \lambda_i \delta f_i \backslash \delta u = 0 \quad t_0 < t < t_f \qquad (98.19)$$

The solutions of Eqs. 98.17 and 98.19 in the above form are difficult to obtain. Nevertheless, based on the above ideas more general theories can be developed such as Pontryagins maximum principle.

An advancement over the calculus variations is the Pontryagins maximum principle, which offers easier solutions and expands the range of applicability to bounded control problems. In its simplest form, this principle may be explained as follows.

Given the system

$$\mathbf{x}_i = f_i[\mathbf{x}, \mathbf{u}, t] \qquad (98.20)$$

and an objective function,

$$J = \sum k_i x_i(t_f) \qquad (98.21)$$

The maximum principle states that if the control vector *u* is optimum, then the Hamiltonian

$$H = \sum p_i f_i \qquad (98.21a)$$

is also maximized with respect to the control vector *u* over the set intervals.

$$p_i = \sum p_j \delta f_j \backslash \delta x_j \qquad (98.21b)$$

$$p_i(t_f) = k_i \qquad (98.21c)$$

where *p* is the adjoint or co-state vector, and

$$\delta H \backslash \delta p_i = f_i[\mathbf{x}, \mathbf{u}, \mathbf{t}] = x_i \qquad (98.21d)$$

$$\delta H \backslash \delta p_i = \sum p_i \delta f_j \backslash \delta x_j = -p_i \qquad (98.21e)$$

The initial values of state vector *x* provide the remaining constants necessary to solve these equations for determining an optimum control vector. This method is also applicable to systems with complex performance indexes.

Another expansion of calculus variations is the Kalman filter. Kalman is essentially a general filtering technique that can be applied to solutions of problems such as optimal estimation, prediction, noise

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filtering, stochastic optimal control, and design of optimal controllers. The method has the advantage of providing estimates of variables in the presence of noise for both stationary and nonstationary processes. Most other types of noise can be treated by this method if they can be translated to Gaussian form. The technique can also be employed in systems with forcing disturbances and containing more than one noise source. Generally, systems that have either quadratic objective functions or uncorrelated Gaussian noise as the input are suitable for this technique.

A means of applying the Kalman filter to a nonlinear system is to find a good estimate of the system and then use it to define a new set of linear **state equations** which approximates the system to linear form at a normal operating point. Then the filter can be applied to the new set of linear equations. Applications of Kalman filter is endless in instrumentation and measurement systems, and other optimal control problems. It can easily be programmed on digital computers.

98.4　Riccati Equation

Some of the models in optimization problems resemble the models of traditional control theory and practice. In these models, the process and control variables are vector-valued and constrained by linear plant equations. The cost functions are in the form of quadratic costs. These type of systems are termed as **linear quadratic equations** systems or LQ systems. The theory and models of LQ systems are well developed in both deterministic and stochastic cases.

Express a quadratic optimal control problem as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{98.22}$$

where　$\mathbf{x}(k)$ = state vector (n-vector)
　　　　$\mathbf{u}(k)$ = control vector (r-vector)
　　　　$\mathbf{A}$ = $n \times n$ nonsingular vector
　　　　$\mathbf{B}$ = $n \times r$ matrix

The aim is to find the optimal control sequence $\mathbf{u}(0)$, $\mathbf{u}(1)$, …, $\mathbf{u}(N-1)$ that minimizes a quadratic performance index

$$J = \frac{1}{2}\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\sum \mathbf{x}^*(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^*(k)\mathbf{R}\mathbf{u}(k) \tag{98.23}$$

where $\mathbf{x}^*$ and $\mathbf{u}^*$ are transposes of $\mathbf{x}$ and $\mathbf{u}$ matrixes, respectively,
　　　$\mathbf{Q}$ = $n \times n$ positive definite or positive semidefinite symmetric matrix
　　　$\mathbf{R}$ = $r \times r$ positive definite or positive semidefinite symmetric matrix
　　　$\mathbf{S}$ = $n \times n$ positive definite or positive semidefinite symmetric matrix

Matrixes $\mathbf{Q}$, $\mathbf{S}$, and $\mathbf{R}$ are selected to weight the relative importance of the performance measures caused by the state vectors $\mathbf{x}(k)$, the final state $\mathbf{x}(N)$, and control vectors $\mathbf{u}(k)$, respectively, for $k = 0, 1, 2, …, N-1$.

The initial state of the system is arbitrary but the final state $\mathbf{x}(N)$ may be fixed. If the final state is fixed, then the term $^1/_2\,\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N)$ may be removed from the performance index and the terminal state $\mathbf{x}_f$ may be imposed. If the final state is not fixed, then the term $^1/_2\,\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N)$ represents the weight of the performance measure to the final state.

There are many different ways of solving the above equations, one of which makes use of the concept of Lagrange multipliers. With the aid of Lagrange multipliers, the performance index may be modified as

$$\mathbf{L} = \frac{1}{2}\mathbf{x}*(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\sum \mathbf{x}*(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}*(k)\mathbf{R}\mathbf{u}(k) + \lambda*(k+1)$$

$$\left[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{x}(k+1)\right] + \left[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{x}(k+1)\right]^* \lambda*(k+1) \tag{98.24}$$

It is known that minimization of the function **L** is equivalent to minimization of performance index **J** under the same constraints.

In order to minimize, **L** needs to be differentiated with respect to vectors $\mathbf{x}(k)$, $\mathbf{u}(k)$, and $\lambda(k)$ and the results set to zero. The partial differentiation of the function **L** with respect to variables gives the following

$$\delta\mathbf{L}/\delta\mathbf{x}(k) = 0 \quad \mathbf{Q}\mathbf{x}(k) + \mathbf{A} * \lambda(k+1) - \lambda(k) = 0 \tag{98.25}$$

$$\delta\mathbf{L}/\delta\mathbf{x}(N) = 0 \quad \mathbf{S}\mathbf{x}(N) - \lambda(N) = 0 \tag{98.26}$$

$$\delta\mathbf{L}/\delta\mathbf{u}(k) = 0 \quad \mathbf{R}\mathbf{u}(k) - \mathbf{B} * \lambda(k+1) = 0 \tag{98.27}$$

and

$$\delta\mathbf{L}/\delta\lambda(k) = 0 \quad \mathbf{A}\mathbf{x}(k-1) + \mathbf{B}\mathbf{u}(k-1) - \mathbf{x}(k) = 0 \tag{98.28}$$

A close inspection of the above formulae indicates that the last equation is simply the state equation for $k = 1, 2, 3, \dots N$. And also, the value of Lagrange multiplier can be determined by Eq. 98.26. The Lagrange multiplier is often termed the covector or the adjoint vector.

Rewriting Eqs. 98.25 and 98.27

$$\lambda(k) = \mathbf{Q}\mathbf{x}(k) + \mathbf{A} * \lambda(k+1) \tag{98.29}$$

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B} * \lambda(k+1) \tag{98.30}$$

the state Eq. 98.22 can be expressed as

$$\mathbf{u}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B} * \lambda(k+1) \tag{98.31}$$

In order to obtain the solution to the minimization problem we need to solve Eqs. 98.29 and 98.31 simultaneously as a two-point boundary-value problem. The solutions of these two equations in this form, and the optimal values of the state vector and the Lagrange multiplier vector will lead to the values of control vector $\mathbf{u}(k)$ which will optimize the open-loop control system. However, for closed-loop control systems the Riccati transformations must be applied.

$$\mathbf{u}(k) = -\mathbf{K}(k)\,\mathbf{x}(k) \tag{98.32}$$

where $\mathbf{K}(k)$ is the $r*n$ feedback matrix. Now the Riccati equation in feedback form can be obtained by assuming that $\lambda(k)$ is written as

$$\lambda(k) = \mathbf{P}(k)\,\mathbf{x}(k) \tag{98.33}$$

where $\mathbf{P}(k)$ is an $n \times n$ matrix. Substituting Eq. 98.33 into 98.29 and 98.31 gives

$$\mathbf{P}(k)\,\mathbf{x}(k) = \mathbf{Q}\mathbf{x}(k) + \mathbf{A} * \mathbf{P}(k+1)\,\mathbf{x}(k+1) \tag{98.34}$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\,\mathbf{R}^{-1}\mathbf{B} * \mathbf{P}(k+1)\,\mathbf{x}(k+1) \tag{98.35}$$

In writing Eqs. 98.34 and 98.35 the Lagrange multiplier $\lambda(k)$ has been eliminated. This is an important step in solving two-point boundary-value problems. By further manipulations it is possible to show that

$$\mathbf{P}(k) = \mathbf{Q} + \mathbf{A} * \mathbf{P}(k+1)\mathbf{A} - \mathbf{A} * \mathbf{P}(k+1)\mathbf{B}\left[\mathbf{R} + \mathbf{B} * \mathbf{P}(k+1)\mathbf{B}\right]^{-1}\mathbf{B} * \mathbf{P}(k+1)\mathbf{A} \qquad (98.36)$$

Equation 98.36 is known as the **Riccati equation**.

From Eqs. 98.26 and 98.33 writing $\lambda(N) = \mathbf{S}\mathbf{x}(N) = \mathbf{P}(N)\,\mathbf{x}(N)$ gives

$$\mathbf{P}(N) = \mathbf{S} \qquad (98.37)$$

Hence, the Riccati equation can be solved backward from $k = N$ to $k = 0$, starting from the known values of $\mathbf{P}(N)$.

The optimal control vector $\mathbf{u}(k)$ can now be calculated from Eqs. 98.29, 98.30, and 98.35

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B} * \lambda(k+1) = -\mathbf{R}^{-1}\mathbf{B} * (\mathbf{A}*)^{-1}\left[\mathbf{P}(k) - \mathbf{Q}\right]\mathbf{x}(k) = -\mathbf{K}(k)\mathbf{x}(k) \qquad (98.38)$$

where

$$\mathbf{K}(k) = -\mathbf{R}^{-1}\mathbf{B} * (\mathbf{A}*)^{-1}\left[\mathbf{P}(k) - \mathbf{Q}\right] \qquad (98.39)$$

It is worthy noting that the optimal control vector may be obtained in slightly different forms by the different manipulations of above equations, such as

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B} * \left[\mathbf{P}^{-1}(k+1) + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}*\right]^{-1}\mathbf{A}\mathbf{x}(k) \qquad (98.40)$$

Equations 98.38 and 98.39 indicate that optimal control law requires feedback of the state vector with time-varying gain $\mathbf{K}(k)$. Figure 98.4 illustrates the optimal control scheme of a system based on the quadratic performance index. In practical applications, the time-varying $\mathbf{K}(k)$ is calculated before the process begins. Once the state matrix $\mathbf{A}$, control matrix $\mathbf{B}$, and weighting matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ are known, the gain $\mathbf{K}(k)$ may be precomputed off-line to be used later. The control vector $\mathbf{u}(k)$ at each stage can be determined immediately by premultiplying the state vector $\mathbf{x}(k)$ by the known gain $\mathbf{K}(k)$.

From the above equations, the minimum value of the performance index can also be calculated. By using the initial values

$$J_{\min} = \frac{1}{2}\mathbf{x} * (0)\mathbf{P}(0)\mathbf{x}(0) \qquad (98.41)$$
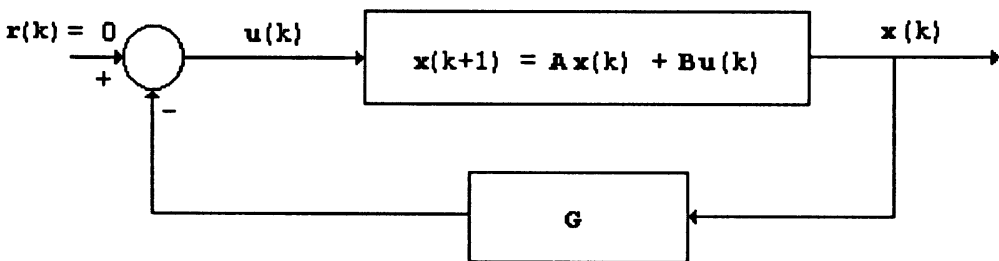
FIGURE 98.4 Optimal control scheme based on quadratic performance index. The control law requires feedback of the state vector with time varying gain $\mathbf{K}(k)$. The gain $\mathbf{K}(k)$ is calculated before the process begins to be used later.

The *steady-state* solutions of Riccati equations are necessary when dealing with time-invariant (steady-state) optimal controls. There are many ways of obtaining steady-state Riccati solutions; an example is given below.

$$\mathbf{P} = \mathbf{Q} + \mathbf{A}*\mathbf{PA} - \mathbf{A}*\mathbf{PB}(\mathbf{R}+\mathbf{B}*\mathbf{PB})^{-1}\mathbf{B}*\mathbf{PA} \tag{98.42}$$

The steady-state value of **K** may be found as

$$\mathbf{K} = (\mathbf{R}+\mathbf{B}*\mathbf{PB})^{-1}\mathbf{B}*\mathbf{PA} \tag{98.43}$$

and the optimal control law for the steady-state operation may be expressed as

$$\mathbf{u}(k) = -(\mathbf{R}+\mathbf{B}*\mathbf{PB})^{-1}\mathbf{B}*\mathbf{PAx}(k) \tag{98.44}$$

This section is presented for discrete-time optimal control systems rather than continuous systems, due to recent widespread use of computers and microprocessors as online and off-line control tools. Since the principles are the same, solutions can easily be extended for continuous time systems with minor modifications.

98.5   State Feedback Matrix

The state feedback method is another design technique that allows the designer to locate the poles of the system wherever they are needed. This type of approach is termed the **pole-placement method**, where the term *pole* refers to the poles of the closed-loop transfer function as in Fig. 98.5. In this method, it is assumed that **state variables** are measurable and are available for feedback. If the state is available, the system is said to be deterministic; the system is noise free and its parameters are fully known. If the state is not available, then methods such as measurement feedback laws or state estimators may be selected.

In many applications, instead of state variables, it is more convenient to use state estimates coming from an observer or Kalman filter. In this case there are three well-known methods available: the certainty equivalent, the separation, and the dual control. Certainty equivalence has considerable advantages over others since it leads to deterministic optimal control laws such Pontryagins principle or the conditional mean of the state such as Kalman filter. It leads to a practical controller, which may be built as a filter and optimal law in cascade.

State feedback controllers are relatively easy to implement. For example, in the pole-placement method the relationship of the feedback control **u** to state **x** for **linear systems** is
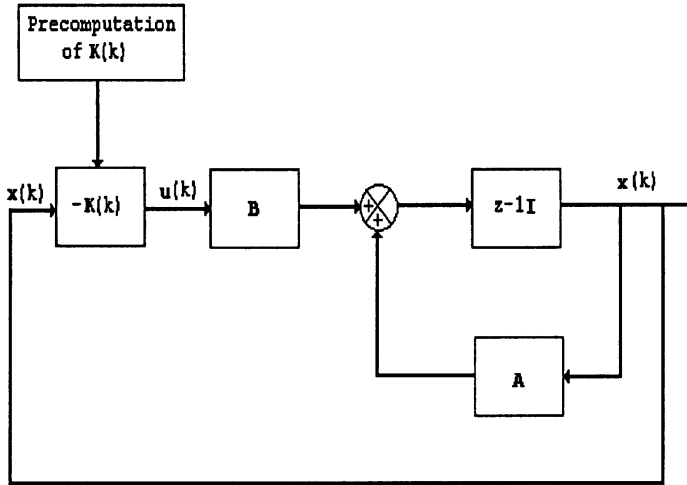
$$\mathbf{u}(k) = -\mathbf{Kx}(k) \tag{98.45}$$

In linear quadratic cases, the gain **K** is time varying:

$$\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k) \tag{98.46}$$

where **K** is $(n \times r)$ feedback matrix. This is correct at least in two cases.

1. In linear systems with no noise, pole-placement controllers, and an observer as the state estimator, the control algorithms for

$$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)$$

$$\mathbf{y}(k) = \mathbf{Cx}(k)$$

FIGURE 98.5 Pole-placement design of closed loop transfer function. This method assumes that state variables are measurable and are available for feedback.

is

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} - \mathbf{GC})\,\hat{\mathbf{x}}(k) + \mathbf{Gy}(k) + \mathbf{Bu}\,(k) \tag{98.47}$$

$$\mathbf{u}(k) = \mathbf{K}\hat{\mathbf{x}}(k) \tag{98.48}$$

where the eigenvalues of $(\mathbf{A} - \mathbf{GC})$ and $(\mathbf{A} + \mathbf{BK})$ are chosen to meet design specifications.

2. In linear systems, influenced by Gaussian white noise, in which the control is optimal according to a quadratic criterion, the system

$$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k) + \mathbf{Gv}(k) \tag{98.49}$$

$$\mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{w}(k) \tag{98.50}$$

has control law of the form

$$\hat{\mathbf{x}}\big(k+1/k\big) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\hat{\mathbf{u}}(k) \tag{98.51}$$

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}\big(k+1/k\big) + \mathbf{G}(k+1)\,\mathbf{y}\,(k+1) - \mathbf{C}\hat{\mathbf{x}}\big(k+1/k\big) \tag{98.52}$$

$$\hat{\mathbf{u}}(k) = \mathbf{K}(k)\,\hat{\mathbf{x}}(k) \tag{98.53}$$

where $\mathbf{K}(k)$ and $\mathbf{G}(k)$ are optimal gains for the deterministic optimal controls and the Kalman filters, respectively.

In practice, not all the state variables are easily accessible, and in general only the outputs of the system are measurable. Therefore, when feedback from the state variables are required in a given design, it is necessary to observe the states from information contained in the output as well as the input variables. The subsystem that performs the observation of the state variables based on the information received from the measurements of inputs and outputs is called the state observer. Figure 98.6 shows the block diagram of such a system.

Suppose a state feedback gain **K** has been selected so that the eigenvalues of

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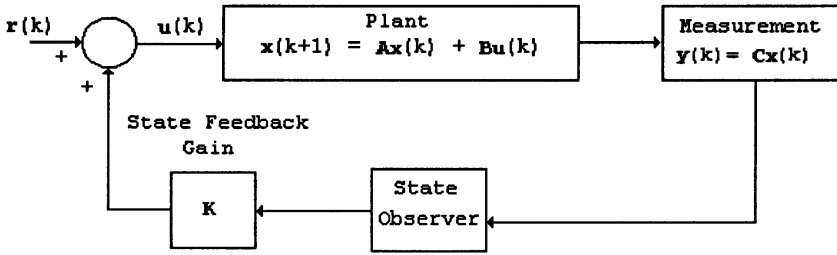
FIGURE 98.6 Use of state observer. When feedbacks from the state variables are required, it may be possible to observe the states from information contained in the output as well as the input variables based on the information received from the measurements.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{98.54}$$

$$\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k) + \mathbf{r}(k) \tag{98.55}$$

are located at $\lambda_1, \lambda_2, \ldots \lambda_n$. Furthermore, assume that gain $\mathbf{G}$ in the identity observer

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} - \mathbf{G}\mathbf{C})\,\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{G}\mathbf{y}(k) \tag{98.56}$$

is chosen such that eigenvalues of the observers are $\mu_1, \mu_2, \ldots \mu_n$.

When the observer state estimate $\mathbf{x}(k)$ is used instead of state $\mathbf{x}(k)$ the resulting system has a state dimension $2\mathbf{n}$, modeled by

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{G}\mathbf{C} & \mathbf{A} - \mathbf{G}\mathbf{C} - \mathbf{B}\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} \mathbf{r}(k) \tag{98.57}$$

A similarity transform

$$\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} \tag{98.58}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \tag{98.59}$$

corresponds to a change of variables to $\mathbf{x}(k)$ and $\mathbf{e}(k) = \mathbf{x}(k) - \mathbf{x}(k)$ converts this to

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{G}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}(k) \tag{98.60}$$

This system has the same eigenvalues as the original because of the nature of the similarity transforms and the eigenvalues are the solution of

$$\det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})\right)\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{G}\mathbf{C})\right) = 0 \tag{98.61}$$

This requires that one of

$$\det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right) = 0 \qquad (98.62)$$

$$\det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{GC})\right) = 0 \qquad (98.63)$$

must hold. These are also the eigenvalues of the design for the pole-placement feedback state controller $\lambda_1, \lambda_2, \ldots \lambda_n$ and of the design of the state observer $\mu1, \mu2, \ldots \mu n$, respectively. This indicates that the use of observer does not move the designed poles from the pole-placement algorithm.

The algorithms for pole placement observer are well known and they are part of many control design packages. Steady-state Kalman gains and optimal control gains are also commonly available in such programs. Codes for Kalman filters and LQ controllers are also easy to write.

Note: For further reading on the topic refer to the sources in references.

Defining Terms

Calculus variations: A technique which can be used for solving linear or nonlinear optimization problems with boundary conditions.

Controllability: Property of a control system such that a determined input takes every state variable from a desired initial state to desired final state.

Controller: A subsystem that assists to achieve the desired output of a plant or process.

Cost function: A function whose value indicates how well the actual performance of the system matches the desired performance.

Covector: Lagrange multiplier.

Hill climbing: A method of determining absolute maxima or minima by using an initial guess point and derivatives of the function.

Kalman Filter: A procedure which provides optimal estimates of state.

Lagrange multiplier: A mathematical expression that modifies the performance index to give an optimal solution.

Linear quadratic equation: Cost functions in quadratic form.

Linear system: A system that possesses the properties of superposition.

Optimal control: A control technique that maximizes or minimizes the cost function of a system.

Optimization: Procedure of obtaining maximum or minimum performance.

Performance index: Cost function.

Pole-placement method: A design technique that makes use of the system closed-loop properties.

Riccati equation: A set of equations that transform control equations to lead to optimal solutions.

State equation: A set of simultaneous, first-order differential equations with n variables leading to solutions of state variables.

State-space model: Mathematical expression of a system that consists of simultaneous, first-order differential equations and an output equation.

State variables: A set of linearly independent system variables so that once the values are set they can determine the value of all system variables later in time.

References

1. Whittle, P., *Optimal Control — Basics and Beyond,* John Wiley & Sons, New York, 1996.
2. Lewis, L.L and V.L. Syrmos, *Optimal Control,* 2nd ed., John Wiley & Sons, New York, 1995.
3. Ogata, K., *Discrete-Time Control Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1987.
4. Kuo, B.C., *Digital Control Systems,* 2nd ed., Harcourt Brace Jovanovich, New York, 1992.

List of Manufacturers

There are no manufacturers of optimal controllers. Most mathematical packages, such as **MATLAB Optimization Toolbox, CAD/CAM** algorithms, the majority of artificial intelligence and neural network software tools, and other simulation and design packages, offer solutions to optimal control problems. Optimization tools are also part of specialized design packages addressing specific applications, such as FPGA, a language-based design, technology-specific optimization; OPTCON, optimal control algorithms for linear stochastic models; DISNEL; MIMO; and so on.